

# Towards Practical Mean Bounds for Small Samples

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# Non-parametric Upper-Confidence Bound

## Assumption.

- The distribution has support  $\mathcal{D} \subseteq [0, 1]$ .

## Definition.

Given a confidence level  $1 - \alpha \in (0, 1)$ , an upper confidence bound  $\mu_{upper}^{1-\alpha}$  has *guaranteed coverage* if, for all sample sizes  $1 \leq n \leq \infty$  and for all distributions  $F$  with support on  $[0, 1]$ , it satisfies

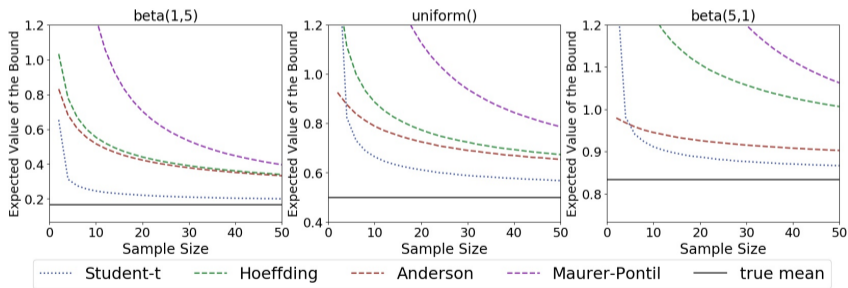
$$Prob_F[\mu \leq \mu_{upper}^{1-\alpha}(X_1, X_2, \dots, X_n)] \geq 1 - \alpha, \quad (1)$$

where  $\mu$  is the mean of the unknown distribution  $F$ .

# Existing Results

Existing confidence bounds for the mean of distributions with support  $\mathcal{D} \subseteq [0, 1]$ .

- With guarantees, but bad performance for small samples (Maurer-Pontil, Hoeffding and Anderson)
- No guarantees (assume Gaussianity), good performance for small samples (Student-t)

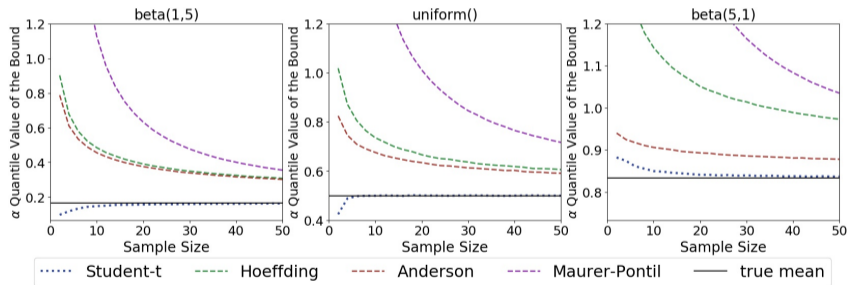


**Figure 1:** The expectation of the confidence bounds for small samples. .

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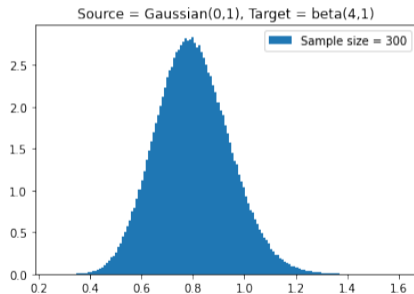


**Figure 2:** The  $\alpha$ -quantile of the confidence bounds for small samples. Student-t does not have guarantee for small sample size.

# Motivations

Consider the problem of finding mean bounds for **small** sample sizes. Examples:

- Phase 1 Clinical Trial.
- Safe Reinforcement Learning: importance-weighted estimators are used to estimate the policy's return.



**Figure 3:** The sample mean of importance-weighted estimator is skewed even for large sample size.

Find an upper confidence bound:

- with guaranteed coverage for distributions on  $[0, 1]$
- with good performance for small sample size

# Our Bound

- Input:
  - A sample  $\mathbf{x} = (x_1, \dots, x_n)$  of size  $n$ .
  - A confidence level  $\alpha$ .
  - A function  $T : [0, 1]^n \rightarrow R$ .
    - Example:  $T(\mathbf{x})$  is the sample mean of  $\mathbf{x}$ .
- Output: upper-confidence bound  $b_T^\alpha(\mathbf{x})$ .
- Our bound is parameterized by a function  $T$ . Function  $T$  defines the order of the bound:  
If  $T(\mathbf{x}) \leq T(\mathbf{y})$  then  $b_T^\alpha(\mathbf{x}) \leq b_T^\alpha(\mathbf{y})$ .
  - Note: Hoeffding's bound and Student-t's bound order the samples by the sample mean.
  - We produce a guaranteed bound with **any** function  $T$ .

A new bound with **guarantee coverage** and **good performance**:

- For any sample size, for any  $\alpha$ , **for any sample  $\mathbf{x}$** , our bound is smaller than or equal to Anderson's [Anderson, 1969], one of the best bound for small sample sizes.
  - For any sample size, for any  $\alpha$ , **for any sample  $\mathbf{x}$** , Anderson's bound is smaller than or equal to Hoeffding [Hoeffding, 1963].
- Therefore for any sample size, for any  $\alpha$ , **for any sample  $\mathbf{x}$** , our bound is smaller than or equal to Hoeffding's.



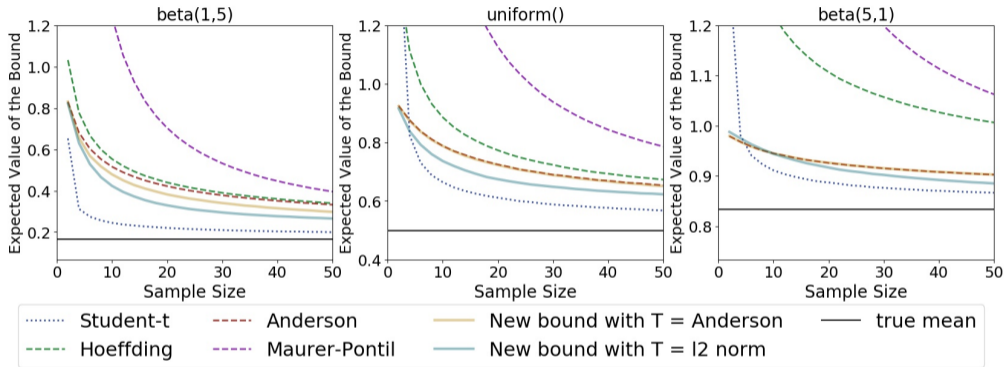
- We use Monte Carlo simulation to compute the bound.
  - The Monte Carlo output is proven to converge to the theoretical value of the bound as the number of Monte Carlo samples increases.
  - Given any error threshold  $\epsilon > 0$ , it is possible to compute the number of Monte Carlo samples required such that with probability at least  $1 - \alpha$ :

$$\mu \leq \widehat{b_T^\alpha(\mathbf{x})} \leq b_T^\alpha(\mathbf{x}) + \epsilon \quad (2)$$

where  $b_T^\alpha(\mathbf{x})$  is the theoretical value of the bound and  $\widehat{b_T^\alpha(\mathbf{x})}$  is the Monte Carlo output.

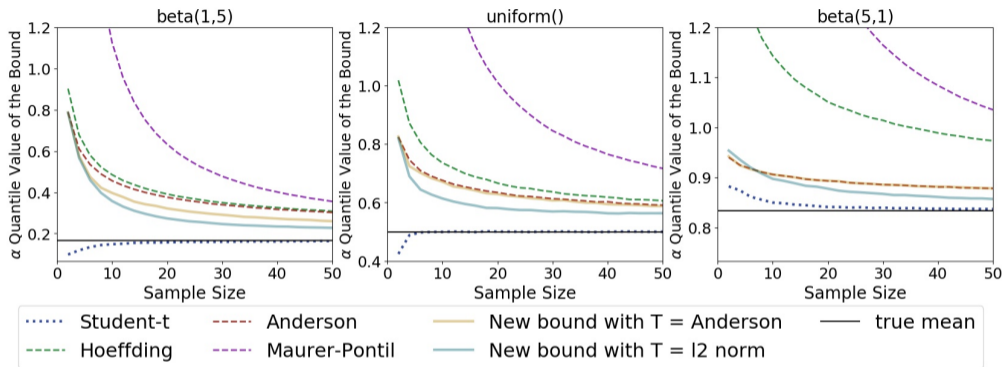
- Computing the bound for one time for  $n = 50$  takes seconds.
- The bound outputs the same value for samples  $\mathbf{x}$  with the same  $T(\mathbf{x})$ . Therefore we can pre-compute a table mapping  $T(\mathbf{x})$  to the value of the bound.

# Simulations i



**Figure 4:** Expected values of the bounds versus sample size  $n$ . For each  $n$ , we sample  $\mathbf{X} \in \mathcal{R}^n$  10,000 times, and take the average of the bound. Our new bound has better performance than Anderson's, Hoeffding's and Maurer and Pontil's.

## Simulations ii



**Figure 5: The  $\alpha$ -quantiles of bound distributions.** If  $\alpha$  portion of the samples' bound is below the true mean, the bound does not have guarantee. For the *uniform(0, 1)* and *beta(1, 5)* distribution, when the sample size is small, Student-t does not have guarantee.

**Thank You!**

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# References

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- T. W. Anderson. Confidence limits for the value of an arbitrary bounded random variable with a continuous distribution function. *Technical Report Number 1, Department of Statistics, Stanford University*, 1969.
- W. Hoeffding. Probability inequalities for sums of bounded random variables. *Journal of the American Statistical Association*, 58(301):13–30, 1963.